

# New Polynomial Case for Efficient Domination in $P_6$ -free Graphs

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**Abstract.** In a graph  $G$ , an *efficient dominating set* is a subset  $D$  of vertices such that  $D$  is an independent set and each vertex outside  $D$  has exactly one neighbor in  $D$ . The EFFICIENT DOMINATING SET problem (EDS) asks for the existence of an efficient dominating set in a given graph  $G$ . The EDS is known to be  $NP$ -complete for  $P_7$ -free graphs, and is known to be polynomial time solvable for  $P_5$ -free graphs. However, the computational complexity of the EDS problem is unknown for  $P_6$ -free graphs. In this paper, we show that the EDS problem can be solved in polynomial time for a subclass of  $P_6$ -free graphs, namely  $(P_6, \text{banner})$ -free graphs.

**Keywords:** Graph algorithms; Domination in graphs; Efficient domination; Perfect code;  $P_6$ -free graphs; Square graph.

## 1 Introduction

Throughout this paper, let  $G = (V, E)$  be a finite, undirected and simple graph. We follow West [17] for standard notations and terminology. If  $\mathcal{F}$  is a family of graphs, a graph  $G$  is said to be  $\mathcal{F}$ -free if it contains no induced subgraph isomorphic to any graph in  $\mathcal{F}$ . Let  $P_t$  denotes the path on  $t$  vertices.

In a graph  $G$ , a subset  $D \subseteq V$  is a *dominating set* if each vertex outside  $D$  has some neighbor in  $D$ . An *efficient dominating set* is a dominating set  $D$  such that  $D$  is an independent set and each vertex outside  $D$  has exactly one neighbor in  $D$ . Efficient dominating sets were introduced by Biggs [1], and are also called perfect codes, perfect dominating sets and independent perfect dominating sets in the literature. We refer to [9] for more information on efficient domination in graphs. The EFFICIENT DOMINATING SET problem (EDS) asks for the existence of an efficient dominating set in a given graph  $G$ . The EDS problem is motivated by various applications such as coding theory and resource allocation in parallel computer networks; see [1, 12].

The EDS is known to be  $NP$ -complete in general, and is known to be  $NP$ -complete for several restricted classes of graphs chordal bipartite graphs [14], planar bipartite graphs [14], and planar graphs with maximum degree three [8].

However, the EDS is solvable in polynomial time for split graphs [5], cocomparability graphs [7], interval graphs [6], circular-arc graphs [6], and for many more classes of graphs (see [4] and the references therein). In particular, EDS is  $NP$ -complete for  $2P_3$ -free chordal graphs [16], and hence EDS remains  $NP$ -complete for  $P_7$ -free graphs. Milanic [15] showed that the EDS is solvable in polynomial time for  $P_5$ -free graphs. Brandstädt and Le [3] showed that the EDS is solvable in polynomial time for  $(E, \text{xNet})$ -free graphs, thereby extending the result on  $P_5$ -free graphs. However, the computational complexity of EDS is unknown for  $P_6$ -free graphs. In [4], Brandstädt et al. showed that the EDS is solvable in polynomial time for  $(P_6, S_{1,2,2})$ -free graphs. Recently, the author showed that EDS is solvable in polynomial time for  $(P_6, S_{1,1,3})$ -free graphs, and  $(P_6, \text{bull})$ -free graphs [11]. We refer to Figure 1 of [4] for the recent complexity status of EDS on several graph classes.

In this paper, we show that the EDS problem can be solved in polynomial time for another subclass of  $P_6$ -free graphs, namely  $(P_6, \text{banner})$ -free graphs, where a *banner* is the graph obtained from a chordless cycle on four vertices by adding a vertex that has exactly one neighbor on the cycle. A banner is also called as  $P$ , 4-apple and  $A_4$  in the literature.

The class of banner-free graphs includes several well studied classes of graphs in the literature such as:  $P_4$ -free graphs (or co-graphs),  $K_{1,3}$ -free graphs (or claw-free graphs), and  $C_4$ -free graphs. Note that from the  $NP$ -completeness result for  $K_{1,3}$ -free graphs [13], it follows that for banner-free graphs, the EDS remains  $NP$ -complete.

If  $G$  is a graph, and if  $S \subseteq V(G)$ , then  $G[S]$  denote the subgraph induced by  $S$  in  $G$ . For any two vertices  $u$  and  $v$  in  $G$ ,  $\text{dist}_G(u, v)$  denote the *distance* between  $u$  and  $v$  in  $G$ . The *square* of a graph  $G = (V, E)$  is the graph  $G^2 = (V, E^2)$  such that  $uv \in E^2$  if and only if  $\text{dist}_G(u, v) \in \{1, 2\}$ .

The following lemma given in [15] (see also [3]) relates the EDS problem on  $G$  and the MAXIMUM WEIGHT INDEPENDENT SET (MWIS) problem on  $G^2$ .

**Lemma 1.** [15] *Let  $G$  be a graph with vertex weight  $w(v)$  equal to the number of neighbors of  $v$  plus one. Then the following statements are equivalent for any subset  $D \subseteq V$ :*

- (i)  $D$  is an efficient dominating set in  $G$ .
- (ii)  $D$  is a minimum weight dominating set in  $G$  with  $\sum_{v \in D} w(v) = |V|$ .
- (iii)  $D$  is a maximum weight independent set in  $G^2$  with  $\sum_{v \in D} w(v) = |V|$ .  $\square$

Thus, the EDS problem on a graph class  $\mathcal{G}$  can be reduced to the MWIS problem on the squares of graphs in  $\mathcal{G}$ .

In Section 2, we show that if  $G$  is a  $(P_6, \text{banner})$ -free graph that has an efficient dominating set, then  $G^2$  is also  $(P_6, \text{banner})$ -free (Theorems 1 and 2). Since MWIS can be solved in polynomial time for  $(P_6, \text{banner})$ -free graphs [2, 10], we deduce that the EDS problem can be solved in polynomial time for  $(P_6, \text{banner})$ -free graphs, by Lemma 1 (Theorem 3).

## 2 EDS in $(P_6, \text{banner})$ -free graphs

In this section, we show that the EDS can be solved efficiently in  $(P_6, \text{banner})$ -free graphs. First, we prove the following:

**Theorem 1** *Let  $G = (V, E)$  be a  $(P_6, \text{banner})$ -free graph. If  $G$  has an efficient dominating set, then  $G^2$  is  $P_6$ -free.*

*Proof of Theorem 1:* Let  $G$  be a  $(P_6, \text{banner})$ -free graph having an efficient dominating set  $D$ , and assume to the contrary that  $G^2$  contains an induced  $P_6$ , say with vertices  $v_1, v_2, v_3, v_4, v_5$  and  $v_6$ , and edges  $v_1v_2, v_2v_3, v_3v_4, v_4v_5$ , and  $v_5v_6$ . Then  $\text{dist}_G(v_1, v_2) \leq 2, \text{dist}_G(v_2, v_3) \leq 2, \text{dist}_G(v_3, v_4) \leq 2, \text{dist}_G(v_4, v_5) \leq 2$ , and  $\text{dist}_G(v_5, v_6) \leq 2$  while  $\text{dist}_G(v_1, v_3) \geq 3, \text{dist}_G(v_1, v_4) \geq 3, \text{dist}_G(v_1, v_5) \geq 3, \text{dist}_G(v_1, v_6) \geq 3, \text{dist}_G(v_2, v_4) \geq 3, \text{dist}_G(v_2, v_5) \geq 3, \text{dist}_G(v_2, v_6) \geq 3, \text{dist}_G(v_3, v_5) \geq 3, \text{dist}_G(v_3, v_6) \geq 3$ , and  $\text{dist}_G(v_4, v_6) \geq 3$ . We often use these distance properties implicitly in the remaining proof.

**Case 1:** Suppose that  $\text{dist}_G(v_5, v_6) = 1$ .

Since  $\text{dist}_G(v_4, v_6) \geq 3$ , we have  $\text{dist}_G(v_4, v_5) = 2$ , and so there exists  $d \in V$  such that  $dv_4, dv_5 \in E$ . If  $\text{dist}_G(v_3, v_4) = 1$ , then since  $\text{dist}_G(v_2, v_4) \geq 3$ , we have  $\text{dist}_G(v_2, v_3) = 2$ , and hence there exists  $x \in V$  such that  $xv_2, xv_3 \in E$ . Now, (i) if  $xd \in E$ , then  $\{v_4, v_3, x, d, v_5\}$  will induce a banner in  $G$ , and (ii) if  $xd \notin E$  then  $\{v_6, v_5, d, v_4, v_3, x\}$  will induce a  $P_6$  in  $G$ , a contradiction.

So, assume that  $\text{dist}_G(v_3, v_4) = 2$ , and hence there exists  $c \in V$  such that  $cv_3, cv_4 \in E$ . Then  $cd \in E$  (otherwise,  $G[\{v_6, v_5, d, v_4, c, v_3\}]$  is a  $P_6$  in  $G$ ). So,  $\text{dist}_G(v_2, v_3) = 2$  (otherwise,  $G[\{v_6, v_5, d, c, v_3, v_2\}]$  is a  $P_6$  in  $G$ ), and hence there exists  $b \in V$  such that  $bv_2, bv_3 \in E$ . Then  $bc \in E$  (otherwise, since  $G[\{v_6, v_5, d, c, v_3, b\}]$  is not an induced  $P_6$  in  $G$ , we have  $bd \in E$ . But, then  $G[\{b, v_3, c, d, v_5\}]$  is a banner in  $G$ , a contradiction), and hence  $bd \in E$  (otherwise,  $G[\{v_6, v_5, d, c, b, v_2\}]$  is a  $P_6$  in  $G$ ). Then  $\text{dist}_G(v_1, v_2) = 2$  (otherwise,  $G[\{v_6, v_5, d, b, v_2, v_1\}]$  is a  $P_6$  in  $G$ ), and hence there exists  $a \in V$  such that  $av_1, av_2 \in E$ . Then  $ab \in E$  (otherwise,  $\{a, v_2, b, d, v_5, v_6\}$  will induce a banner or  $P_6$  in  $G$  according as  $ad \in E$  or  $ad \notin E$  respectively, a contradiction), and hence  $ad \in E$  (otherwise,  $G[\{v_1, a, b, d, v_5, v_6\}]$  is a  $P_6$  in  $G$ ).

Now, we have the following:

**Claim 1**  $a, v_2, b \notin D$ .

*Proof of Claim 1:* We prove the claim by assuming the contrary one by one as follows:

- (i) On the contrary, assume that  $a \in D$ . Then by the definition of  $D$ , we have  $b, v_3, c \notin D$ . So, there exists  $v'_3 \in D$  such that  $v_3v'_3 \in E$ . Then by using the definition of  $D$  and by the distance properties, we see that  $G[\{v'_3, v_3, b, d, v_5, v_6\}]$  is a  $P_6$  in  $G$ , a contradiction. Hence,  $a \notin D$ .
- (ii) On the contrary, assume that  $v_2 \in D$  (or  $b \in D$ ). Then by the definition of  $D$ , we have  $a, v_1 \notin D$ . So, there exists  $v'_1 \in D$  such that  $v_1v'_1 \in E$ . Then

by using the definition of  $D$  and by the distance properties, we see that  $\{v'_1, v_1, a, d, v_5, v_6\}$  will induce a  $P_6$  or banner in  $G$ , a contradiction. Hence,  $v_2, b \notin D$ .  $\blacklozenge$

Since  $a, v_2, b \notin D$  (by Claim 1), there exists  $v'_2 \in D$  such that  $v_2 v'_2 \in E$ . Then  $v'_2 a, v'_2 b \in E$  (otherwise,  $\{v'_2, v_2, a, b, d, v_5, v_6\}$  will induce a banner or a  $P_6$  in  $G$ ).

So,  $v_1 \notin D$ , and hence there exists  $v'_1 \in D$  such that  $v_1 v'_1 \in E$ . Then we show the following:

**Claim 2**  $v'_1 \neq v'_2$  (that is,  $v_1 v'_2 \notin E$ ).

*Proof of Claim 2:* Assume the contrary. Then  $v'_2 d \in E$  (otherwise,  $G[\{v_1, v'_2, b, d, v_5, v_6\}]$  is a  $P_6$  in  $G$ ). Then since  $b, v_3, c \notin D$  (by the definition of  $D$ ), there exists  $v'_3 \in D$  such that  $v_3 v'_3 \in E$ . Also, since  $\text{dist}_G(v_1, v_3) \geq 3$ , we have  $v'_2 \neq v'_3$ . Then by using the definition of  $D$  and by the distance properties, we see that  $G[\{v'_3, v_3, b, d, v_5, v_6\}]$  is a  $P_6$  in  $G$ , a contradiction.  $\blacklozenge$

Now, if  $v'_1 d \in E$ , then  $\{v'_1, v_1, a, d, v_5\}$  will induce a banner in  $G$ , and if  $v'_1 d \notin E$ , then  $\{v'_1, v_1, a, d, v_5, v_6\}$  will induce a  $P_6$  in  $G$ , a contradiction.

**Case 2:** Suppose that  $\text{dist}_G(v_1, v_2) = 2 = \text{dist}_G(v_5, v_6)$ .

Then there exist  $a, e \in V$  such that  $av_1, av_2, ev_5, ev_6 \in E$ . If  $\text{dist}_G(v_2, v_3) = 1$ , then since  $\text{dist}_G(v_2, v_4) \geq 3$ , we have  $\text{dist}_G(v_3, v_4) = 2$ , and hence there exists  $x \in V$  such that  $xv_3, xv_4 \in E$ . But, then  $\{v_1, a, v_2, v_3, x, v_4\}$  will induce either a banner or a  $P_6$  in  $G$ , according as  $ax \in E$  or  $ax \notin E$  respectively, a contradiction. So,  $\text{dist}_G(v_2, v_3) = 2$ . Similarly,  $\text{dist}_G(v_4, v_5) = 2$ . Hence, there exist  $b, d \in V$  such that  $bv_2, bv_3, dv_4, dv_5 \in E$ . Then  $\text{dist}_G(v_3, v_4) = 2$  (otherwise,  $\{v_2, b, v_3, v_4, c, v_5\}$  will induce either a banner or a  $P_6$  in  $G$ , according as  $bd \in E$  or  $bd \notin E$  respectively, a contradiction.) So, there exists  $c \in V$  such that  $cv_3, cv_4 \in E$ . Next, we show the following.

**Claim 3**  $ab, bc, cd, de \in E$ .

*Proof of Claim 3:* We prove the claim by assuming the contrary as follows:

- (i) On the contrary, assume that  $ab \notin E$ . Then  $ac \in E$  (otherwise,  $\{v_1, a, v_2, b, v_3, c, v_4\}$  will induce a banner or  $P_6$  in  $G$ ). Then  $bc \notin E$  (otherwise,  $G[\{v_1, a, v_2, b, c\}]$  is a banner in  $G$ ). Then similar to the case of  $ac \in E$ , we see that  $bd \in E$ , and hence  $ad \notin E$  (otherwise,  $G[\{v_1, a, v_2, b, d\}]$  is a banner in  $G$ ). So,  $cd \in E$  (otherwise,  $G[\{v_1, a, c, v_3, b, d\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v_2, b, v_3, c, d\}]$  is a banner in  $G$ , a contradiction. So,  $ab \in E$ . Similarly,  $de \in E$ .
- (ii) On the contrary, assume that  $bc \notin E$ . Then  $ac \in E$  (otherwise, since  $ab \in E$  (by (i)), we see that  $G[\{v_1, a, b, v_3, c, v_4\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v_1, a, b, v_3, c\}]$  is a banner in  $G$ , a contradiction. So,  $bc \in E$ . Similarly,  $cd \in E$ .  $\blacklozenge$

Then we have the following:

**Claim 4**  $v_2, b, v_5, d \notin D$ .

*Proof of Claim 4 :* (i) On the contrary, assume that  $v_2 \in D$ . Then since  $a, v_1 \notin D$ , there exists  $v'_1 \in D$  such that  $v_1 v'_1 \in E$ . Also, since  $b, v_3, c \notin D$ , there exists  $v'_3 \in D$  such that  $v_3 v'_3 \in E$ . Note that by the distance reason,  $v'_1 \neq v'_3$ . Now,  $G[\{v'_1, v_1, a, b, v_3, v'_3\}]$  is a  $P_6$  in  $G$ , a contradiction. So,  $v_2 \notin D$ . Similarly,  $v_5 \notin D$ . (ii) On the contrary, assume that  $b \in D$ . Then since  $a, v_1 \notin D$ , there exists  $v'_1 \in D$  such that  $v_1 v'_1 \in E$ . Also, since  $c, v_4, d \notin D$ , there exists  $v'_4 \in D$  such that  $v_4 v'_4 \in E$ . Note that by the distance reason,  $v'_1 \neq v'_4$ . Now,  $\{v'_1, v_1, a, b, c, v_4, v'_4\}$  will induce a  $P_6$  in  $G$ , a contradiction. So,  $b \notin D$ . Similarly,  $d \notin D$ .  $\blacklozenge$

**Case 2.1:** Suppose that  $bd \notin E$ .

Then we have the following:

**Claim 5**  $ac, ce \in E$ .

*Proof of Claim 5 :* On the contrary, assume that  $ac \notin E$ . Then  $ad \in E$  (otherwise,  $G[\{v_1, a, b, c, d, v_5\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v_1, a, b, c, d\}]$  is a banner in  $G$ , a contradiction. So,  $ac \in E$ . Similarly,  $ce \in E$ .  $\blacklozenge$

**Claim 6**  $a, e \notin D$ .

*Proof of Claim 6 :* On the contrary, assume that  $a \in D$ . Then since  $b, v_3, c \notin D$ , there exists  $v'_3 \in D$  such that  $v_3 v'_3 \in E$ . Also, since  $d, e \notin D$  and  $v_5 \notin D$  (by Claim 4), there exists  $v'_5 \in D$  such that  $v_5 v'_5 \in E$ . Note that by the distance reason,  $v'_3 \neq v'_5$ . Then since  $v'_3 c, v'_5 c \notin E$  (by the definition of  $D$ ), and since  $v'_3 d \notin E$  (else,  $G[\{v'_3, v_3, c, d, v_5\}]$  is a banner in  $G$ ), we see that  $v'_5 d \in E$  (otherwise,  $G[\{v'_3, v_3, c, d, v_5, v'_5\}]$  is a  $P_6$  in  $G$ ). Again, since  $v'_3 e \notin E$  (else,  $\{v'_3, v_3, c, e, v_6\}$  will induce a banner in  $G$ ), we have  $v'_5 e \in E$  (otherwise,  $G[\{v'_3, v_3, c, e, v_5, v'_5\}]$  is a  $P_6$  in  $G$ ). Hence,  $v_6 \notin D$ , and there exists  $v'_6 \in D$  such that  $v_6 v'_6 \in E$ . Note that  $v'_6 \neq v'_5$  (otherwise,  $G[\{v'_3, v_3, c, d, v'_5, v'_6\}]$  is a  $P_6$  in  $G$ ), and by the distance reason,  $v'_3 \neq v'_6$ . But, then  $G[\{v'_3, v_3, c, e, v_6, v'_6\}]$  is a  $P_6$  in  $G$ , a contradiction. So,  $a \notin D$ . Similarly,  $e \notin D$ .  $\blacklozenge$

By Claims 4 and 6, and by the distance reason, there exist  $v'_2, v'_5 \in D$  ( $v'_2 \neq v'_5$ ) such that  $v_2 v'_2, v_5 v'_5 \in E$ . Then we have the following:

**Claim 7**  $v'_2 b, v'_5 d \in E$ .

*Proof of Claim 7 :* On the contrary, suppose that  $v'_2 b \notin E$ . Then  $v'_2 c \notin E$  (otherwise,  $G[\{v'_2, v_2, b, c, v_4\}]$  is a banner in  $G$ ), and hence  $v'_2 d \in E$  (otherwise,  $G[\{v'_2, v_2, b, c, d, v_5\}]$  is a  $P_6$  in  $G$ ). So,  $v'_5 d \notin E$ . Then by using similar arguments, we deduce that  $v'_5 c \notin E$  and  $v'_5 b \in E$ . Hence,  $v_4, c, d \notin D$ , and there exists  $v'_4 \neq v'_2$  such that  $v_4 v'_4 \in E$ . Then we see that  $v'_4 \neq v'_5$  (otherwise,  $G[\{b, v'_5, v_4, d, v_5\}]$  is a banner in  $G$ ). But, then  $G[\{v'_4, v_4, d, v'_2, v_2, b\}]$  is a  $P_6$  in  $G$ , a contradiction. So,  $v'_2 b \in E$ . Similarly,  $v'_5 d \in E$ .  $\blacklozenge$

Also, we have the following:

**Claim 8**  $v'_2 a, v'_5 e \in E$ .

*Proof of Claim 8:* Assume the contrary. Then  $v'_2c \notin E$  (otherwise,  $G[\{v'_2, v_2, a, c, v_4\}]$  is a banner in  $G$ ). Now, if  $ad \notin E$ , then  $G[\{v'_2, v_2, a, c, d, v_5\}]$  is a  $P_6$  in  $G$ , a contradiction. So, assume that  $ad \in E$ . Now, (i) if  $v_4v'_5 \in E$ , then  $v'_5c \in E$  (otherwise,  $G[\{v'_2, b, c, v_4, v'_5, v_5\}]$  is a  $P_6$  in  $G$ ), and hence  $v'_5a \in E$  (otherwise,  $G[\{v'_2, v_2, a, c, v'_5, v_5\}]$  is a  $P_6$  in  $G$ ). Also,  $v'_2v_3 \notin E$  (otherwise,  $G[\{v_2, v'_2, v_3, c, v'_5, v_5\}]$  is a  $P_6$  in  $G$ ). Thus, since  $b, v_3, c \notin D$ , there exists  $v'_3 \neq v'_2, v'_5$  such that  $v_3v'_3 \in E$ . But, then  $G[\{v'_3, v_3, b, a, v'_5, v_5\}]$  is a  $P_6$  in  $G$ , a contradiction. So, (ii) assume that  $v_4v'_5 \notin E$ . Since  $c, v_4, d \notin D$ , there exists  $v'_4 (\neq v'_5) \in D$  such that  $v_4v'_4 \in E$ . Then  $v'_4a \notin E$  (otherwise,  $G[\{v'_4, v_4, d, a, v_1\}]$  is a banner in  $G$ ). But, then  $G[\{v'_4, v_4, d, a, v'_2, v_2\}]$  is a  $P_6$  in  $G$ , a contradiction. Hence,  $v'_2a \in E$ . By using similar arguments, we can also show that  $v'_5e \in E$ . ♦

Since  $a, v_1 \notin D$ , there exists  $v'_1 \in D$  such that  $v_1v'_1 \in E$ .

Now, if  $v_4v'_5 \in E$ , then  $v'_2c \notin E$  (otherwise,  $G[\{v_2, v'_2, c, v_4, v'_5, v_5\}]$  is a  $P_6$  in  $G$ ), and hence  $v'_5c \in E$  (otherwise,  $G[\{v'_2, b, c, v_4, v'_5, v_5\}]$  is a  $P_6$  in  $G$ ). Then  $v'_2v_3 \notin E$  (otherwise,  $G[\{v_2, v'_2, v_3, c, v'_5, v_5\}]$  is a  $P_6$  in  $G$ ). Thus, since  $b, v_3, c \notin D$ , there exists  $v'_3 \neq v'_2, v'_5$  such that  $v_3v'_3 \in E$ . Note that by the distance reason,  $v'_1 \neq v'_3$ , and  $v'_1 \neq v'_2$  (otherwise,  $G[\{v_1, v'_2, b, c, v'_5, v_5\}]$  is a  $P_6$  in  $G$ ). Now,  $G[\{v'_1, v_1, a, b, v_3, v'_3\}]$  is a  $P_6$  in  $G$ , a contradiction.

So, assume that  $v_4v'_5 \notin E$ . Since  $c, v_4, d \notin D$  and by the distance reasons, there exists  $v'_4 (\neq v'_1, v'_2, v'_5) \in D$  such that  $v_4v'_4 \in E$ . Next, we prove that  $v_1v'_2 \notin E$ . Suppose not. Then  $v'_2c \in E$  (otherwise,  $G[\{v_1, v'_2, b, c, d, v_5\}]$  is a  $P_6$  in  $G$ ). Then  $v'_5v_6 \notin E$  (otherwise,  $G[\{v_1, v'_2, c, d, v'_5, v_6\}]$  is a  $P_6$  in  $G$ ). Since  $e, v_6 \notin D$ , there exists  $v'_6 (\neq v'_5, v'_4, v'_2) \in D$  such that  $v_6v'_6 \in E$ . Then,  $G[\{v_1, v'_2, c, e, v_6, v'_6\}]$  is a  $P_6$  in  $G$ , a contradiction. So,  $v_1v'_2 \notin E$ , and hence there exists  $v'_1 \neq v'_2$ . Then  $v'_1c \notin E$  (otherwise,  $G[\{v'_1, v_1, a, c, v_4\}]$  is a banner in  $G$ ), and hence  $v'_4c \in E$  (otherwise,  $G[\{v'_1, v_1, a, c, v_4, v'_4\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v'_1, v_1, a, d, v_4, v'_4\}]$  or  $G[\{v'_1, v_1, a, c, d, v_5\}]$  is a  $P_6$  in  $G$  according as  $ad \in E$  or  $ad \notin E$  respectively, a contradiction.

**Case 2.2:** Suppose that  $bd \in E$ .

Then either  $ad \in E$  or  $be \in E$  (otherwise,  $G[\{a, b, d, e, v_6\}]$  is a banner in  $G$ , if  $ae \in E$ , and  $G[\{v_1, a, b, d, e, v_6\}]$  is a  $P_6$  in  $G$ , if  $ae \notin E$ , a contradiction). We may assume that  $ad \in E$ . Then we have the following:

**Claim 9**  $a, e \notin D$ .

*Proof of Claim 9:* (i) On the contrary, assume that  $a \in D$ . Then since  $b, v_3, c \notin D$ , there exists  $v'_3 \in D$  such that  $v_3v'_3 \in E$ . Also, since  $d, v_5, e \notin D$ , there exists  $v'_5 \in D$  such that  $v_5v'_5 \in E$ . Note that by the distance reason,  $v'_3 \neq v'_5$ . Now,  $G[\{v'_3, v_3, b, d, v_5, v'_5\}]$  is a  $P_6$  in  $G$ , a contradiction. So,  $a \notin D$ . (ii) On the contrary, assume that  $e \in D$ . Then since  $c, v_4, d \notin D$ , there exists  $v'_4 \in D$  such that  $v_4v'_4 \in E$ . Since  $a \notin D$  (by (i)) and  $v_2, b \notin D$  (by Claim 4), there exists  $v'_2 (\neq v'_4) \in D$  such that  $v_2v'_2 \in E$ . Now, if  $ae \in E$ , then  $v'_2a \notin E$ , and thus  $G[\{v'_2, v_2, a, d, v_4, v'_4\}]$  is a  $P_6$  in  $G$ , a contradiction. So, assume that  $ae \notin E$ . Then  $v'_2a \in E$  (otherwise,  $G[\{v'_2, v_2, a, d, e, v_6\}]$  is a  $P_6$  in  $G$ ), and  $v'_2b \in E$  (otherwise,  $G[\{v'_2, v_2, b, d, v_4, v'_4\}]$  is a  $P_6$  in  $G$ ). So,  $v_1 \notin D$ , and there exists  $v'_1 (\neq v'_4) \in D$  such that  $v_1v'_1 \in E$ . Note that  $v'_1 \neq v'_2$  (otherwise,  $G[\{v_1, v'_2, b, d, v_4, v'_4\}]$  is a

$P_6$  in  $G$ ). But, then  $G[\{v'_1, v_1, a, d, v_4, v'_4\}]$  is a  $P_6$  in  $G$ , a contradiction. Hence,  $e \notin D$ .  $\blacklozenge$

Then by Claims 4 and 9, there exist  $v'_2, v'_5 \in D (v'_2 \neq v'_5)$  such that  $v_2v'_2, v_5v'_5 \in E$ . Then:

**Claim 10**  $v'_2b \in E$ .

*Proof of Claim 10 :* Suppose not. Then  $v'_2c \notin E$  (else,  $G[\{v'_2, v_2, b, c, v_4\}]$  is a banner in  $G$ ). Now, if  $v'_5d \notin E$ , then since  $v'_2d, v'_5b \notin E$  (otherwise, either  $\{v'_2, v_2, b, d, v_5\}$  or  $\{v'_5, v_5, b, d, v_2\}$  will induce a banner in  $G$ ), we see that  $G[\{v'_2, v_2, b, d, v_5, v'_5\}]$  is a  $P_6$  in  $G$ , a contradiction. So, assume that  $v'_5d \in E$ . Then  $v_4v'_5 \in E$  (otherwise, since  $c, v_4, d \notin D$ , there exists  $v'_4 (\neq v'_2, v'_5) \in D$  such that  $v_4v'_4 \in E$ . Then since  $v'_4b \notin E$  (else,  $G[\{v'_4, v_4, d, b, v_2\}]$  is a banner in  $G$ ), we have  $G[\{v'_2, v_2, b, d, v_4, v'_4\}]$  is a  $P_6$  in  $G$ , a contradiction). Then  $v'_5b, v'_5c \in E$  (otherwise,  $\{v'_2, v_2, b, c, v_4, v'_5, v_5\}$  will induce either a  $P_6$  or a banner in  $G$ ).

Now, (i) if  $be \in E$ , then since  $v'_2e \notin E$  (else,  $G[\{v'_2, v_2, b, e, v_6\}]$  is a banner in  $G$ ), we have  $v'_5e \in E$  (otherwise,  $G[\{v'_5, v_5, e, b, v_2\}]$  is a banner in  $G$ ). Then since  $v'_5v_6 \notin E$  (by the distance reason), and since  $e, v_6 \notin D$ , there exists  $v'_6 (\neq v'_2, v'_5) \in D$  such that  $v_6v'_6 \in E$ . But, then  $G[\{v'_6, v_6, e, b, v_2, v'_2\}]$  is a  $P_6$  in  $G$ , a contradiction. So, (ii) assume that  $be \notin E$ . Then  $v'_2e \in E$  (otherwise,  $G[\{v'_2, v_2, b, d, e, v_6\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v_4, v'_5, v_5, e, v'_2, v_2\}]$  is a  $P_6$  in  $G$ , a contradiction. Hence the claim holds.  $\blacklozenge$

Then since  $b, c, v_3 \notin D$ , there exists  $v'_3 (\neq v'_5) \in D$  such that  $v_3v'_3 \in E$ .

**Claim 11**  $v'_5d \in E$ .

*Proof of Claim 11 :* Suppose not. Then  $v'_2a \in E$  (otherwise, since  $v'_2d \notin E$  (else,  $G[\{v'_2, v_2, a, d, v_5\}]$  is a banner in  $G$ ), and since  $v'_5a \notin E$  (else,  $G[\{v'_5, v_5, d, a, v_1\}]$  is a banner in  $G$ ), we have  $G[\{v'_2, v_2, a, d, v_5, v'_5\}]$  is a  $P_6$  in  $G$ , a contradiction). Now, (i) if  $v_1v'_2 \in E$ , then since  $b, v_3, c \notin D$ , there exists  $v'_3 (\neq v'_2, v'_5) \in D$  such that  $v_3v'_3 \in E$ . Then  $v'_3d \in E$  (otherwise,  $G[\{v'_3, v_3, b, d, v_5, v'_5\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v'_3, v_3, b, d, v_5\}]$  is a banner in  $G$ , a contradiction. (ii) So, assume that  $v_1v'_2 \notin E$ , and hence there exists  $v'_1 (\neq v'_2, v'_5) \in D$  such that  $v_1v'_1 \in E$ . Then  $v'_1d \in E$  (otherwise,  $G[\{v'_1, v_1, a, d, v_5, v'_5\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v'_1, v_1, a, d, v_5\}]$  is a banner in  $G$ , a contradiction.  $\blacklozenge$

Then since  $c, d, v_4 \notin D$ , there exists  $v'_4 (\neq v'_2) \in D$  such that  $v_4v'_4 \in E$ .

**Claim 12**  $v'_2a \in E$ .

*Proof of Claim 12 :* Suppose not. Then if  $v'_2v_3 \in E$ , then  $G[\{v_3, v'_2, v_2, a, d, v_5\}]$  is a  $P_6$  in  $G$ , a contradiction. So, assume that  $v'_2 \neq v'_3$ . Then  $v'_5a \notin E$  (else,  $G[\{v'_5, v_5, b, a, v'_5, v'_5\}]$  is a  $P_6$  in  $G$ ). Now, if  $v'_4 \neq v'_5$ , then since  $v'_4a \notin E$  (else,  $G[\{v'_4, v_4, d, a, v_1\}]$  is a banner in  $G$ ), we have  $G[\{v'_2, v_2, a, d, v_4, v'_4\}]$  is a  $P_6$  in  $G$ , a contradiction. So, assume that  $v_4v'_5 \in E$ . Then:

(i) If  $ae \in E$ , then since  $v'_2e \notin E$  (else,  $G[\{v'_2, v_2, a, e, v_6\}]$  is a banner in  $G$ ), we have  $v'_5e \in E$  (otherwise,  $G[\{v'_2, v_2, a, e, v_5, v'_5\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v_4, v'_5, e, a, v_2, v'_2\}]$  is a  $P_6$  in  $G$ , a contradiction.

(ii) If  $ae \notin E$ , then since  $v'_2e \in E$  (else,  $G[\{v'_2, v_2, a, d, e, v_6\}]$  is a  $P_6$  in  $G$ ), we have  $be \in E$  (otherwise,  $G[\{v'_2, b, d, e, v_6\}]$  is a banner in  $G$ ). But, then  $G[\{v'_3, v_3, b, e, v_5, v'_5\}]$  is a  $P_6$  in  $G$ , a contradiction.

So the claim holds.  $\blacklozenge$

Hence,  $v_1 \notin D$ , and thus there exists  $v_1v'_1 \in E$ .

**Claim 13**  $v'_1 \neq v'_2$ .

*Proof of Claim 13 :* Suppose not. assume that  $v_1v'_2 \in E$ . Then by the distance reason,  $v'_3 \neq v'_2$ . Then: (i) If  $be \in E$ , then since  $v'_3e \notin E$  (else,  $G[\{v'_3, v_3, b, e, v_6\}]$  is a banner in  $G$ ), we have  $v'_5e \in E$  (else,  $G[\{v'_3, v_3, b, e, v_5, v'_5\}]$  is a  $P_6$ ), and hence  $v'_5v_6 \notin E$  (otherwise,  $G[\{v_1, v'_2, b, d, v'_5, v_6\}]$  is a  $P_6$ ). Since  $e, v_6 \notin D$ , there exists  $v'_6 (\neq v'_3, v'_5) \in D$  such that  $v_6v'_6 \in E$ . But, then  $G[\{v'_6, v_6, e, b, v_3, v'_3\}]$  is a  $P_6$  in  $G$ , a contradiction. (ii) If  $be \notin E$ , then  $v'_2e \in E$  (else,  $G[\{v_1, v'_2, b, d, e, v_6\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v'_2, b, d, e, v_6\}]$  is a banner in  $G$ , a contradiction.  $\blacklozenge$

Then  $v'_3 = v'_2$  and  $v'_4 = v'_5$  (otherwise, either  $G[\{v'_1, v_1, a, b, v_3, v'_3\}]$  or  $G[\{v'_1, v_1, a, d, v_4, v'_4\}]$  is a  $P_6$  in  $G$ ). That is,  $v_3v'_2, v_4v'_5 \in E$ . Then: (i) If  $ae \in E$ , then since  $v'_1e \notin E$  (else,  $G[\{v'_1, v_1, a, e, v_6\}]$  is a banner in  $G$ ), we have  $v'_5e \in E$  (otherwise,  $G[\{v'_1, v_1, a, e, v_5, v'_5\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v'_1, v_1, a, e, v'_5, v_4\}]$  is a  $P_6$ , a contradiction. (ii) If  $ae \notin E$ , then  $v'_2e \in E$  (else,  $G[\{v_3, v'_2, a, d, e, v_6\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v_4, v'_5, v_5, e, v'_2, v_2\}]$  is a  $P_6$  in  $G$ , a contradiction.

Since the other cases are symmetric, we have proved the theorem.  $\square$

Next, we prove the following:

**Theorem 2** *Let  $G = (V, E)$  be a  $(P_6, \text{banner})$ -free graph. If  $G$  has an efficient dominating set, then  $G^2$  is banner-free.*

*Proof of Theorem 2 :* Let  $G$  be a  $(P_6, \text{banner})$ -free graph having an efficient dominating set  $D$ , and assume to the contrary that  $G^2$  contains an induced banner, say with vertices  $v_1, v_2, v_3, v_4$ , and  $v_5$ , and edges  $v_1v_2, v_2v_3, v_3v_4, v_4v_1$ , and  $v_3v_5$ . Then  $\text{dist}_G(v_1, v_2) \leq 2, \text{dist}_G(v_2, v_3) \leq 2, \text{dist}_G(v_3, v_4) \leq 2, \text{dist}_G(v_4, v_1) \leq 2$ , and  $\text{dist}_G(v_3, v_5) \leq 2$ , while  $\text{dist}_G(v_1, v_3) \geq 3, \text{dist}_G(v_1, v_5) \geq 3, \text{dist}_G(v_2, v_4) \geq 3, \text{dist}_G(v_2, v_5) \geq 3$ , and  $\text{dist}_G(v_4, v_5) \geq 3$ . We often use these distance properties implicitly in the remaining proof.

**Case 1:** Suppose that  $\text{dist}_G(v_3, v_5) = 1$ .

Then since  $\text{dist}_G(v_2, v_5) \geq 3, \text{dist}_G(v_2, v_3) = 2$ . Again, since  $\text{dist}_G(v_4, v_5) \geq 3$ , we have  $\text{dist}_G(v_3, v_4) = 2$ . So, there exist vertices  $a$  and  $b$  in  $V$  such that  $av_2, av_3, bv_3, bv_4 \in E$ . Since  $\text{dist}_G(v_2, v_4) \geq 3$ , at least one of  $\text{dist}_G(v_1, v_4), \text{dist}_G(v_1, v_2)$  is equal to two. We may assume (wlog.) that  $\text{dist}_G(v_1, v_2) = 2$ . Hence, there exists  $c \in V$  such that  $cv_1, cv_2 \in E$ . Then  $ac \in E$  (otherwise,  $\{v_5, v_3, a, v_2, c, v_1\}$  will induce a  $P_6$  in  $G$ ). Then  $\text{dist}_G(v_1, v_4) = 2$  (otherwise,  $\{v_5, v_3, a, c, v_1, v_4\}$  will induce a  $P_6$  in  $G$ ). Hence, there exists  $d \in V$  such that  $dv_1, dv_4 \in E$ . As earlier,  $bd \in E$ , and hence  $cd \in E$  (otherwise, either  $G[\{v_5, v_3, b, d, v_1, c\}]$  is a  $P_6$  in  $G$  or  $G[\{v_3, b, d, v_1, c\}]$  is a banner in  $G$  according as  $bc \notin E$  or  $bc \in E$  respectively, a contradiction). Then  $bc, ad \in E$  (otherwise, either  $G[\{v_5, v_3, b, d, c, v_2\}]$  is a  $P_6$  in  $G$  or  $G[\{v_5, v_3, a, c, d, v_4\}]$  is a  $P_6$  in  $G$ ),



and hence  $ab \in E$  (otherwise,  $G[\{v_5, v_3, b, d, a\}]$  is a banner in  $G$ ). Then we have the following:

**Claim 14**  $v_2, v_4, a, b, c, d \notin D$ .

*Proof of Claim 14 :* We prove the claim by assuming the contrary one by one as follows:

- (i) On the contrary, assume that  $v_2 \in D$ . Then by the definition of  $D$ , we have  $d, c, v_1 \notin D$ . So, there exists  $v'_1 \in D$  such that  $v_1 v'_1 \in E$ . Then by the distance properties and by the definition of  $D$ , we see that  $G[\{v'_1, v_1, c, a, v_3, v_5\}]$  is a  $P_6$  in  $G$ , a contradiction. Hence,  $v_2 \notin D$ . Similarly,  $v_4 \notin D$ .
- (ii) On the contrary, assume that  $a \in D$ . Then by the definition of  $D$ , we have  $d, b, v_4 \notin D$ . So, there exists  $v'_4 \in D$  such that  $v_4 v'_4 \in E$ . Then by the definition of  $D$  and by using the distance properties, we see that  $G[\{v'_4, v_4, d, a, v_3, v_5\}]$  is a  $P_6$  in  $G$ , a contradiction. Hence,  $a \notin D$ . Similarly,  $b \notin D$ .
- (iii) On the contrary, assume that  $c \in D$ . Then since  $b, d, v_4 \notin D$ , there  $v'_4 \in D$  such that  $v_4 v'_4 \in E$ . Then by distance reasons and by the definition of  $D$ , we have  $v'_4 v_3 \in E$  (otherwise,  $G[\{v'_4, v_4, d, a, v_3, v_5\}]$  is a  $P_6$  in  $G$ ), and hence  $G[\{v'_4, v_4, b, v_3, c\}]$  is a banner in  $G$ , a contradiction. Hence,  $c \notin D$ . Similarly,  $d \notin D$ . ♦

So, there exist  $v'_2, v'_4 \in D$  such that  $v_2 v'_2, v_4 v'_4 \in E$ . Note that by the distance reason, we have  $v'_2 \neq v'_4$ . Now, we prove the following:

**Claim 15**  $v'_2 c, v'_4 d \in E$ .

*Proof of Claim 15 :* On the contrary, assume that  $v'_2 c \notin E$ . Then  $v'_2 b \notin E$  (else,  $G[\{v'_2, v_2, c, b, v_4\}]$  is a banner in  $G$ ), and  $v'_2 v_3 \in E$  (else,  $G[\{v'_2, v_2, c, b, v_3, v_5\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v_5, v_3, v'_2, v_2, c, v_1\}]$  is a  $P_6$  in  $G$ , a contradiction. So,  $v'_2 c \in E$ . Similarly,  $v'_4 d \in E$ . ♦

So,  $v_1 \notin D$ , and hence there exists  $v'_1 \in D$  such that  $v_1 v'_1 \in E$ . Then, we show the following:

**Claim 16**  $v'_1 \neq v'_2, v'_4$ .

*Proof of Claim 16 :* Assume the contrary, and assume that  $v'_1 = v'_2$ . That is,  $v_1 v'_2 \in E$ . Then since  $v'_2 v_3 \notin E$  (by the distance reason), we have  $v'_2 b \in E$  (otherwise,  $G[\{v_5, v_3, b, d, v_1, v'_2\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v'_2, v_1, d, b, v'_4\}]$  is a banner in  $G$ , a contradiction. So,  $v'_1 \neq v'_2$ . Similarly,  $v'_1 \neq v'_4$ . ♦

Now, since  $v'_1 c \notin E$  (by the definition of  $D$ ) and since  $v'_1 a \notin E$  (else,  $G[\{v'_1, v_1, c, a, v_3\}]$  is a banner in  $G$ ), we see that by the distance properties,  $G[\{v'_1, v_1, c, a, v_3, v_5\}]$  is a  $P_6$  in  $G$ , a contradiction.

**Case 2:** Suppose that  $\text{dist}_G(v_3, v_5) = 2$ .

So, there exists  $a \in V$  such that  $av_3, av_5 \in E$ . Then  $\text{dist}_G(v_2, v_3) = 2 = \text{dist}_G(v_3, v_4)$  (otherwise, if  $\text{dist}_G(v_2, v_3) = 1$ , then  $\text{dist}_G(v_1, v_2) = 1$  (else, there

exists  $x \in V$  such that  $xv_1, xv_2 \in E$ , and hence  $\{v_5, a, v_3, v_2, x, v_1\}$  will induce either a  $P_6$  or a banner in  $G$ , a contradiction). Hence, a contradiction to the fact that  $\text{dist}_G(v_1, v_3) \geq 3$ . A similar contradiction can be arrived if  $\text{dist}_G(v_3, v_4) = 1$ . So, there exist  $b, c \in V$  such that  $bv_2, bv_3, cv_3, cv_4 \in E$ .

Since  $\text{dist}_G(v_2, v_4) \geq 3$ , we may assume that  $\text{dist}_G(v_1, v_4) = 2$ , and there exists  $d \in V$  such that  $dv_1, dv_4 \in E$ . Then  $\text{dist}_G(v_1, v_2) = 2$  (otherwise,  $\{v_4, d, v_1, v_2, b, v_3\}$  will induce either a  $P_6$  or a banner in  $G$ ), and hence there exists  $e \in V$  such that  $ev_1, ev_2 \in E$ .

**Claim 17**  $ac, ab, bc, cd, de, be \in E$ .

*Proof of Claim 17 :* (i) On the contrary, assume that  $ac \notin E$ . Then  $ad \in E$  (otherwise,  $\{v_5, a, v_3, c, v_4, d, v_1\}$  will induce a  $P_6$  in  $G$ ), and hence  $cd \notin E$  (otherwise,  $G[\{v_5, a, v_3, c, d\}]$  is a banner in  $G$ ). Now, if  $ab \notin E$ , then  $bd \notin E$  (else,  $G[\{v_5, a, v_3, b, d\}]$  is a banner in  $G$ ). Then  $cb \in E$  (otherwise,  $G[\{v_1, d, v_4, c, v_3, b\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v_5, a, d, v_4, c, b\}]$  is a  $P_6$  in  $G$ , a contradiction. So, assume that  $ab \in E$ , then  $be \in E$  (otherwise,  $\{v_5, a, b, v_2, e, v_1\}$  will induce either a  $P_6$  or banner in  $G$ ). Then  $bc \in E$  (otherwise,  $\{v_1, e, b, v_3, c, v_4\}$  will induce either a  $P_6$  or banner in  $G$ ), and hence  $bd \notin E$  (otherwise,  $G[\{d, v_4, c, b, v_2\}]$  is a banner in  $G$ ). But, then  $G[\{v_1, d, v_4, c, b, v_2\}]$  is a  $P_6$  in  $G$ , a contradiction. So,  $ac \in E$ . Similarly,  $ab \in E$ .

(ii) On the contrary, assume that  $cd \notin E$ . Then,  $\{v_5, a, c, v_4, d, v_1\}$  will induce either a  $P_6$  or banner in  $G$ , a contradiction. So,  $cd \in E$ . Similarly,  $be \in E$ .

(iii) On the contrary, assume that  $bc \notin E$ . Then,  $\{v_4, c, v_3, b, e, v_1\}$  will induce either a  $P_6$  or banner in  $G$ , a contradiction. So,  $bc \in E$ . Again, by using similar arguments, we see that  $de \in E$ . ♦

Next, we show the following:

**Claim 18**  $v_2, v_4, b, c, d, e \notin D$ .

*Proof of Claim 18 :* We prove the claim by assuming the contrary one by one as follows:

- (i) On the contrary, assume that  $v_2 \in D$ . Then since  $d, e, v_1 \notin D$ , there exists  $v'_1 \in D$  such that  $v_1v'_1 \in E$ . Also, since  $a, b, c, v_3 \notin D$ , there exists  $v'_3 \in D$  such that  $v_3v'_3 \in E$ . By the distance reason,  $v'_1 \neq v'_3$ . But, then  $G[\{v'_1, v_1, e, b, v_3, v'_3\}]$  is a  $P_6$  in  $G$ , a contradiction. Hence,  $v_2 \notin D$ . Similarly,  $v_4 \notin D$ .
- (ii) On the contrary, assume that  $c \in D$ . Then since  $a, v_5 \notin D$ , there exists  $v'_5 \in D$  such that  $v_5v'_5 \in E$ . Also, since  $e, b, v_2 \notin D$ , there exists  $v'_2 \in D$  such that  $v_2v'_2 \in E$ . By the distance reason,  $v'_2 \neq v'_5$ . But, then  $G[\{v'_5, v_5, a, b, v_2, v'_2\}]$  is a  $P_6$  in  $G$ , a contradiction. Hence,  $c \notin D$ . Similarly,  $b \notin D$ .
- (iii) On the contrary, assume that  $d \in D$ . Then since  $e, b, v_2 \notin D$ , there exists  $v'_2 \in D$  such that  $v_2v'_2 \in E$ . Also, since  $a, b, c, v_3 \notin D$ , there exists  $v'_3 \in D$  such that  $v_3v'_3 \in E$ . Then,  $v'_2 \neq v'_3$  (otherwise,  $G[\{v_2, v'_2(=v'_3), v_3, c, d, v_1\}]$  is a  $P_6$  in  $G$ ). But, then  $\{v'_3, v_3, c, d, e, v_2, v'_2\}$  will induce a  $P_6$  in  $G$ , a contradiction. Hence,  $d \notin D$ . Similarly,  $e \notin D$ .

By (i), (ii) and (iii), we see that Claim 18 is proved.  $\blacklozenge$

So, there exist  $v'_2, v'_4 \in D$  such that  $v_2v'_2, v_4v'_4 \in E$ . Note that by the distance reason,  $v'_2 \neq v'_4$ .

**Claim 19**  $v_1 \notin D$ .

*Proof of Claim 19 :* If not, then  $G[\{v'_4, v_4, d, e, v_2, v'_2\}]$  is a  $P_6$  in  $G$ , a contradiction. Hence,  $v_1 \notin D$ .  $\blacklozenge$

Since  $d, e, v_1 \notin D$  (by Claims 18 and 19), there exists  $v'_1 \in D$  such that  $v_1v'_1 \in E$ . Then:

**Claim 20**  $v'_1 \neq v'_2$  and  $v'_1 \neq v'_4$ .

*Proof of Claim 20 :* On the contrary, suppose that  $v'_1 = v'_2$ . That is,  $v_1v'_2 \in E$ . Then either  $v'_4d \in E$  or  $v'_2d \in E$  (not both) (otherwise,  $G[\{v'_4, v_4, d, v_1, v'_2, v_2\}]$  is a  $P_6$  in  $G$ ). Now, if  $v'_4d \in E$  and  $v'_2d \notin E$ , then by using the distance properties, we see that  $v'_2c \in E$  (otherwise,  $G[\{v_3, c, d, v_1, v'_2, v_2\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v_1, v'_2, c, d, v'_4\}]$  is a banner in  $G$ , a contradiction. So, assume that  $v'_2d \in E$  and  $v'_4d \notin E$ . Then  $v'_2e \in E$  (otherwise,  $G[\{e, v_2, v'_2, d, v_4\}]$  is a banner in  $G$ ). Then  $v'_2b \in E$  (otherwise, since  $bd \in E$  (else,  $G[\{v_4, d, v'_2, v_2, b, v_3\}]$  is a  $P_6$  in  $G$ ), we see that  $G[\{v_4, d, b, v_2, v'_2\}]$  is a banner in  $G$ ). Now, if  $v'_4v_3 \in E$ , then  $G[\{v_4, v'_4, v_3, b, v'_2, v_1\}]$  is a  $P_6$  in  $G$ , a contradiction. So, assume that  $v'_4v_3 \notin E$ . Since  $v_3 \notin D$ , there exists  $v'_3 (\neq v'_2, v'_4) \in D$  such that  $v_3v'_3 \in E$ . Now,  $\{v'_3, v_3, b, e, d, v_4, v'_4\}$  will induce a  $P_6$  in  $G$ , a contradiction. Hence the claim holds.  $\blacklozenge$

Then either  $v'_4d \in E$  or  $v'_2e \in E$  (otherwise, since  $G[\{v'_4, v_4, d, e, v_2, v'_2\}]$  is not a  $P_6$  in  $G$ , either  $v'_4e \in E$  or  $v'_2d \in E$ . Then either  $G[\{v_4, v'_4, d, e, v_2\}]$  or  $G[\{v_2, v'_2, d, e, v_4\}]$  is a banner in  $G$ , a contradiction). We may assume that  $v'_4d \in E$ . Then  $v'_1c \notin E$  (otherwise,  $G[\{v'_1, v_1, d, c, v_3\}]$  is a banner in  $G$ ).

Now, if  $ad \notin E$ , then  $v'_1a \in E$  (otherwise,  $G[\{v'_1, v_1, d, c, a, v_5\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{v_5, a, v'_1, v_1, d, v'_4\}]$  is a  $P_6$  in  $G$ , a contradiction.

So, assume that  $ad \in E$ . Then  $v'_1a \notin E$  (else,  $G[\{v'_1, v_1, d, a, v_5\}]$  is a banner in  $G$ ). Now, we prove the following.

**Claim 21**  $bd \in E$ .

*Proof of Claim 21 :* If not, then  $v'_1b \in E$  (otherwise,  $G[\{v'_1, v_1, d, c, b, v_2\}]$  is a  $P_6$  in  $G$ ). But, then  $G[\{d, v_1, v'_1, b, v_2, v'_2\}]$  is a  $P_6$  in  $G$ , a contradiction. Hence,  $bd \in E$ .  $\blacklozenge$

Then since  $v'_1b \notin E$  (otherwise,  $G[\{v'_1, v_1, d, b, v_3\}]$  is a banner in  $G$ ), we have  $v'_2b \in E$  (otherwise,  $G[\{v'_1, v_1, d, b, v_2, v'_2\}]$  is a  $P_6$  in  $G$ ). So, since  $a, b, c, v_3 \notin D$ , there exists  $v'_3 (\neq v'_1) \in D$  such that  $v_3v'_3 \in E$ . Then:

**Claim 22**  $v'_3 \neq v'_2, v'_4$ . That is,  $v_3v'_2, v_3v'_4 \notin E$ .

*Proof of Claim 22 :* Suppose not. If  $v_3v'_2 \in E$ , then  $\{v'_1, v_1, d, a, v_3, v'_2, v_2\}$  will induce a  $P_6$  in  $G$ , and if  $v_3v'_4 \in E$ , then  $G[\{v_3, v'_4, d, b, v_2\}]$  is a banner in  $G$ , a contradiction. So, the claim holds.  $\blacklozenge$

Hence,  $G[\{v'_1, v_1, d, b, v_3, v'_3\}]$  is a  $P_6$  in  $G$ , a contradiction.  $\square$

**Theorem 3** *The EDS can be solved in polynomial time for  $(P_6, \text{banner})$ -free graphs.*

*Proof of Theorem 3:* Since the MWIS problem in  $(P_6, \text{banner})$ -free graphs can be solved in polynomial time [2, 10], the theorem follows by Theorems 1 and 2, and Lemma 1.  $\square$

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